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The New Logic

BY

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THE rapid development of physics, the result of observations made and ideas introduced within the last few decades, has brought about a change in the whole system of physical concepts. This fact is common knowledge, and has already attracted the attention of philosophers. It is less well known that geometry too has had its crises, and undergone a reconstruction. For centuries, so-called "geometrical intuition" was used as a method of proof. In geometrical demonstrations, certain steps were allowed because they were "self-evident," because the correctness of the conclusion was "shown by the adjoined diagram," etc. A crisis occurred in geometry because such intuition proved to be untrustworthy. Many of the propositions regarded as self-evident or based upon the consideration of diagrams turned out to be false. And so Euclidean geometry was reconstructed by methods free from all intuitive elements and strictly logical in nature. Moreover, for more than a century, various other geometries have been devised purely as logical constructions. Since they start with assumptions different from those of Euclid, and lead to conclusions partly in contradiction with his theorems, they are called "non-Euclidean." Nevertheless, each one of these geometries is a closed system of

* *Translator's Note:* This paper is a revised version of a lecture delivered in 1932 and published in "Krise und Neuaufbau in den exakten Wissenschaften." (Deuticke, Vienna, 1933.)

propositions exempt from contradiction. Recently, some of them have even found application in physics.

As far as experience and intuition are concerned, anyone, even though he be unfamiliar with the details of physics and geometry, can well understand that new empirical discoveries may overthrow the most venerable of ancient theories, and that intuition, if it ventures too far afield, may be forced to retire from positions erroneously occupied. One subject, however, is generally supposed to be unchanging and unshakable. That subject is logic. Hence, to anyone who is not an adept in the field, the recent active discussion of crisis and reconstruction within the realm of logic may seem not only strange but incomprehensible.

As a matter of fact, for two thousand years, logic has been the most conservative of all the sciences. Aristotle, who is considered the father of the subject, assumes as a starting point that every proposition ascribes a predicate to some subject. The propositions are classified on the one hand as affirmative or negative, on the other hand as universal or particular. "All cats are mammals" is universal and affirmative; "Some mammals are cats" is particular and affirmative; "Some mammals are not cats" is particular and negative; "No cat is a fish" is universal and negative. According to Aristotle, all inference consists in deducing a third proposition from two propositions of the given form. For example, from the premises that all cats are mammals and all mammals are vertebrates, it follows that all cats are vertebrates. From the premises that all cats are mammals and no mammal is a fish, it follows that no cat is a fish. All the inferences which Aristotle considered possible he arranged in three figures divided into altogether fourteen varieties. In the Middle Ages, these modes of inference were expanded to four figures including all together nineteen varieties, and were designated by the names Barbara, Celarent, etc. If, from the premises that to every (resp. no) M belongs the predicate P, and that to every S belongs the predicate M, it is inferred that to every (resp. no) S belongs the predicate P, then the inference is said to be according to the mode Barbara (resp. Celarent). Moreover, the three principles of identity, contradiction and excluded middle, which were later called the fundamental principles of logic, were also formulated by Aristotle, curiously enough not in his logic but in his metaphysics. The kernel of his system, the above mentioned logic of subject-predicate propositions, was essentially all that was regarded as pure logic for two thousand years thereafter.

In the Middle Ages, it is true, the scholastics undertook a number of important logical investigations. But more recently, and particularly during the period of the Enlightenment, it became customary to regard the Labors of the scholars in the Middle Ages as hair-splitting; their work in logic was thus forgotten and was replaced by less fundamental considerations. Similarly, Leibniz's views on logic, which were far in advance of their time, also remained without direct effect. It was perfectly clear to Leibniz that a mere treatment of subject-predicate propositions was insufficient, and must be supplemented by a logic of relations. Furthermore, he treated the logical principles and their mutual relations more systematically than his predecessors, and devised the project of a "lingua characteristica," which should permit all scientific propositions to be stated in precise form, and of a "calculus ratiocinator," which should contain and treat mathematically all the methods of inference. But the clearest evidence that Leibniz's views, like the works of the scholastics, found no echo, is the famous dictum of Kant in the introduction to the second edition of the "Critique of Pure Reason." "That logic has trodden this sure path since the earliest times, can be seen from the fact that, since Aristotle, it has not been obliged to take a single backward step.—A further noteworthy fact is that until the present it has been able to take no step forward, and so seems apparently to be finished and complete." And in Kant's "Logic" occurs the statement "The logic of to-day is derived from Aristotle's Analysis.—Moreover, since Aristotle's time, logic has not gained much in content, and from its very nature it cannot. For Aristotle omitted no item of reason."

The first of the crises in logic was caused by defects in this Aristotelion system which were brought to light by mathematics. When confidence in geometrical intuition had been shaken, there came a purely logical reconstruction of geometry. This consisted of a complete enumeration of all the hypotheses (axioms) concerning fundamental concepts not explicitly defined, and a deduction of the whole system of theorems from these axioms by strictly logical methods without any appeal to experience, diagrams, self-evidence or intuition. An analogous purely logical construction of arithmetic followed. This trend in mathematics naturally carried within a search for clarity in regard to all the principles of inference used in the deduction of mathematical propositions from axioms. And in the course of this search, it became evident that the old logic was utterly insufficient for the demands of modern mathematics in respect to both precision and completeness. It was therefore chiefly mathematicians who undertook and carried out the desired reconstruction of logic. A short sketch of the chief stages in this reconstruction (which is called logistics) will next be given; but it should be understood that this logistics, although much more far-reaching than the Aristotelian logic, is to-day counted as part of the classical content of logic. It will appear that the real problems of the new logic only begin where logistics stops.

The first step in the reconstruction was the development of the so-called calculus of classes (also called algebra of logic) particularly by Boole, Pierce and Schröder in the second half of the last century.[1] The Aristotelian logic, as stated, deals mostly with the question: If the relations of two classes to a third class are known, what can be said about the relations of the two classes to each other? For example, if the relations of the class of all cats and the class of all vertebrates to the class of all mammals are known, what can be said of the relations of the class of all cats to the class of all vertebrates? The calculus of classes is more extensive; for it is a systematic theory of the relations of any number of classes. Aristotle only investigates inclusions. He asks whether of two classes one is entirely or partly contained in

[1] See Boole, "The Mathematical Analysis of Logic," Cambridge (1847), and "An Investigation of the Laws of Thought," London (1854) and Chicago (1916); Schröder, "Vorlesungen über die Algebra der Logik," 3 vols. Teubner, Leipzig (1890–1910) and "Abriss der Logik," ibid. (1909–1910). See also the first chapters of Lewis and Langford "Symbolic Logic," The Century Company, New York.

the other, or whether one is entirely or partly outside the other. But the calculus of classes investigates many other relations between classes besides inclusion, and undertakes many other operations with classes. Among other things, it treats systematically of the "join" and "intersection" of two classes A and B—that is the class of all elements in either A or B as well as the class of those in both A and B. In contradistinction to Aristotle, the null class (containing no member) is also considered. For instance, the intersection of the class of all cats and the class of all fish is the null class. The calculus of classes, starting from a certain few propositions, gives a systematic treatment of the relations of all classes. Among its theorems, there are nineteen which correspond to the Aristotelian scholastic modes of inference.[2]

But in the calculus of classes, these nineteen theorems are not the only ones; moreover they occupy in it no distinguished position. They are not among its initial propositions (in the first place because the whole of this calculus can be based on much fewer than nineteen propositions), and so they are not necessary for the founding of a systematic calculus of classes. But even if it were desired to assume so needlessly large a number of initial propositions, the particular nineteen Aristotelian ones would not be sufficient for the construction of the whole calculus of classes. This calculus is therefore a definite advance over the old logic of class inclusion.

A second step leads further. The calculus of classes is a theory deduced from a few initial propositions concerning the relations of classes, just as Euclidean geometry is deduced from a few initial propositions (axioms) concerning the relations of points,

[2] According to Ladd Franklin, these can moreover be combined into a single theorem, namely the theorem that for the three classes A, B and C, it cannot be simultaneously true that the intersection of A and B is the null class, the intersection of not-B and C is the null class, and the intersection of A and C is not the null class. In the case of the mode Barbara, the premises are: All M is P and all S is M. That is, the intersection of the class M with the class not-P is the null class, and the intersection of the class S with the class not-M is the null class. Then, according to the Ladd Franklin formula, it is impossible that the intersection of the classes S and not-P should not be the null class. In other words, the intersection of the class S and the class not-P is the null class. That is, all S is P, and the assertion of the mode Barbara is thereby proved.

lines, and planes.[3] The calculus of classes is thus a special mathematical theory; it is, however, far from containing the whole of logic, for logic does not confine itself to the consideration of classes. If further propositions are deduced from any set of propositions, this deduction is called logical inference. The subject matter of this inference is, however, not classes but propositions. And yet logic is expected to deal with the rules of inference. The second step in the expansion of logistics, which goes back principally to Pierce and Schröder, was therefore the development of a calculus of propositions. This calculus teaches how propositions are combined by words like "and," "or," "not" and similar particles.[4] If p and q are two propositions, then the propositions (not-p), (not-q), (p or q) and (p and q) are also considered. Particularly important is the relation "q or not-p" which is briefly expressed in logic by the words "p implies q" or "q is implied by p." If q is true, then the proposition "q or not-p" is surely true, whether p is true or false. According to the given terminology, a true proposition q is therefore implied by every proposition. If p is false, then not-p is true, and the proposition "q or not-p" is surely true whether q is true or false. Thus a false proposition p implies every proposition q.

Of special importance among the propositions composed of several statements connected by logical particles are those which are true in all cases, whether the component statements are true or false. For example, the proposition "p or not-p" (It is raining or it is not raining.) is true, whether the component statement "p" (It is raining.) is true or false. At Wittgenstein's suggestion,

[3] The calculus of classes is not only methodologically the same as an axiomatic geometry. It can, in fact, as I have on occasion remarked, be brought formally under one heading with elementary geometrical theories. That is, there exists a theory including elementary geometry and the calculus of classes. From this theory, each of the two more special theories may be obtained by specific additional axioms. Cf. Jahresbericht der deutschen Mathematiker-Vereinigung, *37*, 309 (1928), and Annals of Mathematics, *37*, 456 (1936).

[4] More detailed expositions of the calculus of propositions as well as of the calculus of functions are to be found in Frege, "Begriffsschrift," Halle (1879); Whitehead-Russell, "Principia Mathematica," Cambridge University Press (1925); Hilbert-Ackermann, "Grundzüge der Theoretischen Logik, Springer, Berlin (1928); Carnap, "Abriss der Logistik," Springer, Vienna (1929).

such complex propositions which are always true are called tautologies. The calculus of propositions is concerned with the setting up of tautologies, and it follows from Frege's work that all tautologies can be deduced from a few simple tautologies.

It may be thought that the calculus of propositions treats only those transformations which result from the applications of the three Aristotelian principles (identity, contradiction and excluded middle) to propositions. Such, however, is not the fact. More correctly, the three Aristotelian principles play a rôle in the calculus of propositions similar to that played by the Aristotelian modes of inference in the calculus of classes. That is, the principles of identity, contradiction and excluded middle occur among the theorems of the calculus of propositions, but they are not the only theorems of this calculus; and moreover they enjoy in it no specially distinguished position. In particular, they do not appear among those propositions actually chosen as initial propositions for this calculus; thus they are not a necessary part of its foundation. Moreover, they are not sufficient for the deduction of the whole of it. The calculus of propositions is thus a decided advance over the old logic.

Since the whole calculus of propositions can be deduced from certain simple initial propositions, and since, on the other hand, the principles of this calculus are supposed to be the principles of logical deduction, it might perhaps be suspected that the calculus in question is founded on some sort of reasoning in a circle. Frege, however, carefully avoided this danger. The initial propositions in question are merely certain propositional formulae—in fact tautologies. According to Lukasiewicz, the three following which involve the two undefined concepts "not" and "implies" may be chosen for this purpose.

(1) p implies (not-p implies q).

(2) (not-p implies p) implies p.

(3) (p implies q) implies [(q implies r) implies (p implies r)].

Of these, the first corresponds to the previously mentioned condition that a false proposition implies every proposition. The second corresponds in a certain sense to the law of double negation; for, starting with "or," the proposition "not-p implies p" stands for "p or not-not-p," and the second formula stands for "not-p or (p or not-not-p)." The third formula is closely related to the syllogism; if q is implied by p, and r is implied by q, then r is implied by q.

The calculus of propositions is the totality of those propositional formulae which can be obtained from the three initial formulae by the use of certain formative rules. These are simply the following. First, in the initial formulae or in formulae already obtained from them, the symbols p, q and r may be replaced by other and possibly complex propositional symbols. For example, if, in the first initial formula, "r implies s" is substituted for p, there is obtained the formula "(r implies s) implies [not (r implies s) implies q]" which, according to the first formative rule, is to be taken into the calculus of propositions. Second, if the proposition p and the proposition "p implies q" are two formulae in the calculus of propositions, then q is also admitted as a formula of this calculus. It is thus clear that the calculus in question and the inference used for its systematic development are kept carefully separate. The calculus consists of formulae; the development of the calculus is accomplished by the use of two rules for the construction of formulae. Both the initial formulae and the two formative rules are precisely defined. Obviously the two latter are considerably simpler than the usual rules of logical inference.

But even a combination of the calculus of propositions with the calculus of classes does not exhaust the content of logic. Most common propositions, particularly those of mathematics, besides employing such words as "and," "or," "not" or "implies," contain other logical particals, especially "all," "some" or "there are." The exact rules for dealing with propositions containing these so-called logical quantifiers forms a third chapter of the newer logic which, since the work of Pierce and Frege, has taken its place beside the calculi of classes and of propositions; it is called the calculus of functions. The historical origin of this name is as follows. Besides propositions, some of which are true like "This charcoal is black," and some of which are false like "This charcoal is red," there are also word combinations like "x is black" which are not propositions, but which become propositions only when the name of a definite individual within a certain field is used to replace the symbol x, or when the combination is preceded by a quantifier. Such word combinations are called propositional functions. For example, the propositional function "x is black" becomes a true proposition if "this charcoal" is substituted for x. It becomes a false proposition if "this lime" is substituted for x. When mankind is chosen as the range of x, the proposition becomes false if the quantifier "all" is placed before x, for then it becomes the false proposition "All men are black." Finally, if the quantifier "some," is placed before x, the proposition becomes the true existence proposition "Some men are black." Since the rules for logical operations on general and existence propositions are deduced from the theory of propositional functions, the study of operations with such propositions is called the calculus of functions.

The calculi of classes and of propositions together with the hitherto considered portions of the calculus of functions may be interpreted as mere refinements on the old logic—if the word "refinement" is used in a very broad sense. But the fourth step in the development of the new logic is undoubtedly an extension of the content of the subject. The impetus to this step also came from mathematics; for the propositions which are the subject of mathematics, and which were first expressed by Peano in a general and rigorous symbolism, are only rarely statements about the membership of individuals in classes or about inclusions between classes. Neither are they often propositions consisting of statements about classes connected by the words "and," "or," "not," "implies," "all" or "some." Most mathematical propositions deal rather with relations, as Leibniz already recognized. The proposition "3 is less than r" states a relation between two numbers; the proposition "If, on a straight line, the point q lies between the points p and r, then r does not lie between p and q" is a general statement about a relation between the members of triples of points on straight lines. A logic useful to mathematicians must above all treat of relations.

A predicate corresponds to a class, namely the class of all those things which have that predicate. For instance, the predicate black corresponds to the class of all black things. Similarly, a relation between two things (called a dyadic relation) corresponds to a class of pairs of things, namely the class of all those pairs of things in which the first member of the pair has the given relation to the second. For example, the relation "less than" corresponds to the class of all pairs in which the first member is less than the second. The extension of logic which treats propositions on relationship along with subject-predicate propositions can thus be characterized by the fact that besides classes of individuals, it investigates classes of pairs of individuals, classes of triples of individuals, etc. This subject also starts with certain initial formulae, and, by the help of a few exactly defined rules, deduces from them a system.

Although logic is thus greatly increased in content, it is still insufficient to account for all the conclusions drawn in modern mathematics. In order to follow the newest mathematics, particularly the theories of real numbers and of sets, a fifth step had to be taken. It was necessary to create an expanded calculus of functions which deals with classes of all sorts of classes of individuals, with classes of classes of classes, etc. This extremely important extension of logic is not only necessary to account for all of modern mathematics, but according to Russell, is also sufficient (when taken in connection with the portions of logic previously discussed) to serve as a basis for all of that science. That such is the fact will next be shown in a short sketch.[5]

Once in possession of this expanded calculus of functions, it is possible to define when two classes A and B are equinumerate or of the same power. Since Georg Cantor's day, they have been thus designated when they can be placed in one to one correspondence, with one another—that is, when to every member of the class A, a member of the class B can be assigned in such fashion that each member of B is assigned to just one member of

[5] An easily comprehensible exposition of the logical foundations of mathematics is Russell's "Introduction to Mathematical Philosophy," Allen and Unwin, London (1920).

A. For example, the classes of points and crosses here shown

+ + + + +

. . . . .

* * * *

are said to be of equal power because the class of points can be placed in one to one correspondence with the class of crosses by assigning each point to the cross directly above it. On the other hand, the classes of points and stars are not of equal power, for, no matter how a star is assigned to each point, there will still be a star which is assigned to more than one point. Children and savages count low-numbered classes by bringing the members of the class to be counted into one-to-one correspondence with the members of a class of the fingers of their hands. This fact apparently accounts for the importance of the number ten in the common number system.

It should not be imagined that the concept of equality of number presupposes a concept of number or an operation of counting. Without counting how many auditors or seats there are in a lecture room, the number of auditors and the number of seats may be shown to be the same by ascertaining that each seat is occupied and that each auditor has a seat; by these facts, the one-to-one correspondence necessary and sufficient for equality of number is established between the class of all auditors and the class of all seats. In the theory here outlined the concept of number depends on the concept of equality of number, for a number is defined as the class of all classes which are equinumerate with a given class. For instance, five is defined as the class of all those classes (such as the classes of crosses and points shown above) which are equinumerate with the class of the fingers of one hand. But Cantor went further. For each infinite set S, he combined into a class all those sets which are of the same power as S. One of his great discoveries on which he founded the theory of sets is that various infinite numbers can be obtained in this way, for not every infinite set is of the same power as every other infinite set.

Among all the numbers thus obtained, the finite or natural numbers (1, 2, 3 . . .) are defined simply by the application of the logical concepts. The definition is based essentially on the fact that the important principle of complete induction holds for these numbers. This principle is that any proposition which holds for the number one, and which, if it holds for any natural number n, also holds for the succeeding number $n + 1$, holds for every natural number. A proposition valid for all natural numbers and proved by the principle of complete induction is that every natural number can be expressed in one and only one way as a product of prime numbers. This principle of complete induction has shown itself to be one of the most important methods of proof in arithmetic.

Within the field of the natural numbers, addition and multiplication can be carried on without restriction, but subtraction and division cannot always be performed. That is, to every pair of natural numbers a and b there corresponds a natural number $a + b$ as their sum and one ab as their product, whereas natural numbers $a - b$ and a/b exist only if b is respectively smaller than or a divisor of a. In order to make subtraction possible for any two numbers, a new calculus of pairs of natural numbers is introduced. The pairs (a, b) and (c, d) are called equal if and only if $a + d = b + c$. Addition and multiplication of such pairs are defined by the formulae

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \times (c, d) = (ac + bd, ad + bc)$$

Development of this theory shows that the properties of addition and multiplication are the same among such pairs as in the field of the natural numbers. The pairs (0, 0), (1, 0), (2, 0) . . . act in all respects like the natural numbers 0, 1, 2 . . . . The pairs (0, 1), (0, 2), (0, 3) . . . may be briefly designated as -1, -2, -3 . . . since $(a, 0) + (0, a) = (a, a) = (0, 0)$ . . . an equation which corresponds to the equation $a + (-a) = 0$.

The calculus is constructed so that the integer g is defined as the set of all pairs of natural numbers (a, b) such that $b + g = a$ if g is positive, and $b + (-g) = a$ if g is negative. In an analogous way, rational numbers (fractions) are introduced as pairs of integers by another new calculus which permits unlimited performance of addition, subtraction, multiplication and division (except division by 0). Briefly stated, integers are introduced as differences between natural numbers; rational numbers are introduced as quotients between integers. These rational numbers can be ordered according to their magnitudes. The quotient g/h is called greater, equal to or less than the quotient g'/h' if and only if gh' is respectively greater, equal to or smaller than g'h. Between any two fractions a/b and c/d a fraction of inter mediate size for example (a + c)/(b + d) can be interpolated.

Nevertheless there remain gaps between the fractions. These are associated with the fact that, in the field of fractions, addition, subtraction, multiplication and division (except division by 0) can be carried on without restriction, but other operations, for example the extraction of roots cannot always be performed. It is easy to show, for instance, that there is no fraction a/b such that its square a^2/b^2 is equal to 2, a fact which is also expressed by the statement that $\sqrt{2}$ is not a rational number. However there are fractions with squares arbitrarily close to but less than 2 and fractions with squares arbitrarily close to but greater than 2. Between these two classes of fractions (the lower class consisting of all those with squares less than 2, and the upper class consisting of all those with squares greater than 2) there is a gap in the class of all fractions, for a fraction with a square equal to 2 does not exist. In order to fill out such gaps, and in order to make possible a large number of other operations (like extraction of roots, solution of all equations of odd order, etc.) the real numbers are introduced as a certain kind of division of the class of all fractions into two sub-classes. Those divisions corresponding to gaps in the class of fractions are called irrational numbers. As soon as all the real numbers have been introduced, it is at last possible to define logically the notion of limit so fundamental for all higher mathematics, and thus to define logically the notions of continuity and the differentiation or integration of functions, as well as the concept of n-dimensional space which underlies analytic geometry. This last is defined as the class of all n-tuples of real numbers. By another new calculus with pairs of real numbers, the complex numbers are introduced. Then, according to the fundamental theory of algebra, all equations (in particular the equation $x^2 = -1$) become solvable, and every equation of the n-th degree has n roots.

In a word, after logic has been provided with the expanded calculus of functions, mathematics, as Russell says, becomes a part of logic. This assertion is not by any means an arbitrary extension of the word logic to include all of mathematics. The historical development has just been described. All the conclusions drawn by modern mathematics were treated just as the most primitive conclusions were treated in the calculus of propositions. In other words, all these conclusions were derived from certain initial formulae by the help of certain formative rules. Thus there were obtained initial formulae which not only permit the deduction of mathematical conclusions, but also suffice for the derivation of all mathematics.

Such was the culmination of logistics, which at this point contained all that was needed. After two thousand years of petrification, logic had, in less than half a century, been entirely reconstructed by the mathematicians; and in the year 1900 an article on logic might have terminated with this happy result.

About the turn of the century, however, there came an entirely unexpected repercussion of the worse kind, and a second crisis in logic, a crisis in logistics occurred; for the newly introduced method of unlimited operation with classes and classes of classes led to nothing less than an antinomy. Now an inner contradiction is unbearable even in the special theory of a particular science. For logic, however, the appearance of a contradiction in its structure is catastrophic.

Before explaining the newly discovered antinomy, it should be mentioned that paradoxes had appeared even in ancient logic. The one about the liar is well known. It was called the Cretan inference, because the Cretans were supposed to be particularly mendacious. In ancient times, the statement "All Cretans are liars" in the mouth of a Cretan was regarded as a logical paradox. The best way to illustrate this sort of paradox with modern precision is to write on a board the following statement:

"The statement on this board is false."

Nothing further is written on the board, and the given statement is examined to see whether it is true or false. It will be proved to be neither. Assuming that the statement is true, it follows that the statement on the board is false, in contradiction with the assumption that the statement is true. Assuming that the statement is false, since this is precisely that the statement on the board, it follows that the statement on the board is true, in contradiction with the assumption that the statement is false. Thus the assumption that the statement is false leads to the conclusion that it is true; and the assumption that the statement is true leads to the conclusion that it is false. In other words, the assumption of either truth or falsehood for the statement leads to a contradiction. Hence, the statement is neither true nor false, which is of course paradoxical, for, according to the principle of the excluded middle, every statement is either true or false, and any third possibility is excluded.

As a matter of fact, precise analysis shows that the words "statement on this board" play an essential rôle in the paradox in which they appear. These words are not of a logical nature and, for this reason, the above and related paradoxes are (at Ramsay's suggestion) distinguished by a special name from purely logical antinomies and called epistemological paradoxes. According to a recent research by Tarski,[6] what such paradoxes really show is that a self-consistent use or correct definition of the term "true proposition" which is in harmony both with the laws of logic and with every-day language is impossible.

The severe crisis in logistics was brought on in 1901 by Russell's discovery (following some related results obtained by Burali-Forti) of a purely logical antinomy. Such an antinomy is one in which appear only the concepts of the logical calculus of functions, particularly the concept of class. If M is the class of all men, then

(a) Every member of M is a man.
(b) Every man is a member of M.

[6] Tarski, "Der Wahrheitsbegriff in den formalistischen Sprachen," Studia Philosophica, Lespoli (1935). This important paper contains the first successful treatment of a classical problem in philosophy by means of modern logic.

The class M itself is not a man but a class of men, and therefore, according to (a) does not occur among the members of M. Similarly the class of all triangles in a plane is not a triangle and therefore does not appear among its own members. As these and many other examples show, there certainly exist classes which do not occur among their own members. On the other hand, if N is the class of all non-men, then

(a) Every member of N is not a man.

(b) Everything that is not a man is a member of N.

As N itself is not a man but a class of non-men, according to (b), N occurs among the members of N. Another example of a class which occurs among its own member is the class of all classes. If this class is called K, then

(a) Every member of K is a class.

(b) Every class is a member of K.

As K itself is a class (namely the class of all classes), according to (b), K occurs among the members of K. Now let L be the class of all classes which do not occur among their own members. Then

(a) Every class which is a member of L does not occur among its own members. (For example, the classes K and N just mentioned are not members of L.)

(b) Every class which does not occur among its own members is a member of L. (For example, the class of all men M and the class of all triangles in a plane are members of L.)

The question at issue is whether or not the class L occurs among its own members. First: It is impossible for L to be a member of L. For if L were a member of the class L, then L would contain as a member a class (namely L) which occurred among its own members, whereas according to (a) every class which is a member of L does not occur among its own members. Second: It is impossible for L not to be a member of L. For if L were not a member of L, then L would be a class which did not occur among its own members and still was not a member of L, whereas according to (b) every class which does not occur among its own members is a member of L. Hence it is impossible for L to be a member of

L, and it is impossible for L not to be a member of L. This, however, is an antinomy, for, according to the principle of the excluded middle, every proposition is either true or false and, as has been shown, the proposition "L is a member of L" is neither true nor false.

The effect of Russell's discovery was enormous. Most of what has since been done in logic has been more or less influenced by the fear of antinomies. It will be shown that the importance of the three methods pursued by modern logic far transcends that of the antinomies themselves, but historically the latter have been the most important stimulus for the recent developments.

The first means of surmounting the crisis was found by Russell,[7] the discoverer of the purely logical antinomy which had caused the trouble. His solution is as follows: Every logical construction which occurs in mathematics must have as a starting point a certain field of individuals. Besides these individuals, there come under consideration classes of individuals, which should not be confused with the individuals themselves. Furthermore, there are classes of classes of such individuals (called classes of the second type) which are not to be confused with classes of individuals (called classes of the first type). In general, for every natural number n, there are classes of the n-th type, and all these classes of different type must be carefully distinguished. Particularly, in speaking of all classes, it must always be indicated whether all classes of the first, second or n-th type are meant. A class which contains as members classes of different types must not be formed. The class of all classes mentioned above is a concept which cannot occur if this prohibition is respected. It does not belong in the hierarchy of types. The class of all classes which do not occur among their own members is similarly forbidden, and so this concept which opens the door to antinomies is excluded by the type theory.

As a matter of fact, up to now, no antinomies have been discovered which are not excluded if the rules of the type theory are observed. But although this theory forbids certain concepts which have not led and probably will not lead to contradictions,

[7] Cf. particularly Russell's "Introduction" cited in (4).

it gives no assurance that all possible but still undiscovered antinomies are excluded by it. For this reason, Poincaré, under similar circumstances, spoke of a shepherd who, to protect his flock from wolves, built a fence around it, without, however, being sure that he had not enclosed a wolf within the fence.

A second program which has been followed since a logical crisis was precipitated by the discovery of antinomies consists in the development of a method which had already been vigorously pursued by the mathematician Kronecker in the 80's of the last century, even before the antimonies had been found. It has been called by his followers intuitionism.[8] This name is borrowed from philosophy where it is attributed to the methods initiated by Bergson. As will appear, there are really points of contact between the ideas of the two schools, though there are mathematician-intuitionists who sometimes deny this assertion.

Whereas Russell calls mathematics a part of logic, Kronecker takes his stand on the primacy of mathematics as opposed to logic. According to him, mathematical construction is fundamental. "The whole numbers were created by God; everything else is human handiwork." In particular, logical inference, if unaccompanied by mathematical constructions, can lead to mathematically false conclusions. Above all, Kronecker opposes indirect proofs. In these, it is proved that the non-existence of an entity with the given properties involves a contradiction. Hence the existence of an entity with the given properties is inferred, although no method for the construction of such an entity need be given. An example may illustrate this attitude. Suppose it were possible to deduce a contradiction from the assumption (called Fermat's hypothesis) that for every integer $n > 2$ and for every triple of integers x, y and z, the inequality $x^n + y^n \neq z^n$ subsists. The classical

[8] I gave an account of the historical development of this school of thought with many literature references in an article "Der Intuitionismus" Blätter f. deutsche Philosophie, *4*, 311 (1930). More recent results are summed up by Heyting in "Ergebnisse d. Mathematik u. ihrer Grenzgebiete," vol. 3, Springer, Berlin (1934). An account of the discussions of foundation questions together with many literature references is to be found in Fränkel's "Mengenlehre," 3rd ed., Springer, Berlin (1928).

mathematician would express this fact in the following words: There exist four integers n (> 2), x, y and z such that $x^n + y^n = z^n$. Kronecker, on the other hand, would enunciate this existence theorem only if four such integers had been found, or at least if there had been given a method which would permit the finding of such integers in a finite number of steps. If it were proved that integers of the desired kind existed among those with 1000 digits, then such integers could certainly be found if at all the quadruples those with 1000 digits were examined successively to see whether or not they satisfied the equality $x^n + y^n = z^n$. This process would, it is true, be so tedious that no man could carry it out, but nevertheless it would lead to the goal in a finite number of steps. On the other hand, the mere proof of the impossibility of the non-existence of integers of the desired kind generally furnishes no procedure for the discovery of such integers. In this latter case, Kronecker denies that the existence of an integer of the desired kind has been proved.

Poincaré directed his attacks against what he called impredicative definitions—definitions of an entity which refer to classes in which the entity to be defined belongs. The class of all classes is a typical impredicative concept. As a matter of fact, the concepts of largest number and of an upper limit to a set of numbers (concepts which are important for all higher mathematics) are also impredicative. Hence Weyl in his book "The Continuum" (1918) concluded that these concepts are to be discarded save when the entities in question can be determined operationally —that is, when they can be fixed by a definition which is not impredicative.

Early in this century, the question whether the so-called "axiom of choice" formulated by Zermelo was to be accepted or rejected played a great rôle in the discussions of the Paris school. In fact this question caused more argument than the "axiom of comprehension" which permits arbitrary subsets of any set to be formed, although this latter axiom leads, I believe, to much greater difficulties than does the former, and in certain constructions even condenses all the difficulties into one. Complete induction, on the other hand, is regarded as a valid conclusion by the Paris school. Poincaré considered it a synthetic, a priori judgment in the sense of Kant.

During the last two decades, Brouwer has followed out the consequences of Kronecker's theses, and has shown how much of modern mathematics rests on indirect proofs of existence, and falls with these proofs. Logic, according to him, gives only the rules of the language for mathematical inference about finite systems. To apply it to infinite systems (sequences, sets, etc.) is meaningless. In particular, the principle of the excluded middle, the source of the indirect proofs of existence, is not to be applied to infinite systems. Brouwer takes over Kronecker's restriction on the use of the words "there exists"; they are to mean "there can be defined in a finite number of steps." And so (as long as Fermat's hypothesis has been neither proved nor disproved) he denies the validity of the alternative: Either for every quadruple of integers n (> 2), x, y and z the inequality $x^n + y^n \neq z^n$ holds or there exist four integers n (> 2), x, y and z such that $x^n + y^n = z^n$. Only when the Fermat problem shall have been solved (that is when it shall have been proved either that if $n > 2$, $x^n + y^n \neq z^n$ for all integers, or a quadruple shall have been found such that $x^n + y^n = z^n$) . . . only then will it be seen a posteriori that one of the two possibilities maintained by the alternative actually occurs. As long as the Fermat hypothesis is neither proved nor disproved, not only is the solution unknown, but the very solubility of the problem is uncertain, so that a priori even the alternative cannot be asserted. On the other hand, it cannot be maintained a priori that the Fermat problem or any given problem is insoluble.

According to Brouwer, his own constructions, complete induction in particular, are based on an "original intuition" which, he says, also yields a concept of set. Investigations in the theory of sets which go beyond this concept he discards as meaningless. For instance, he adopts Bergson's view that continuity cannot logically be treated in the classic fashion as a completed aggregate of points.

This intuitionist program requires examination. Does it, first of all, avoid all danger of antinomies? Intuitionists claim that their pure constructions can "obviously" never lead to a contradiction. As a matter of fact, a strong feeling of assurance attaches to those parts of arithmetic which are admitted by intuitionists; this feeling, however, lessens when it comes to their theory of sets. But anyone who, in his search for consistency, requires something more than feelings, or allusions to obviousness and intuition finds no help in intuitionism.

Moreover, the feeling that the intuitionist calculus of propositions and theory of numbers are more secure against contradiction than the corresponding classic theories can be proved to be illusory. The proof was recently established by Gödel. The erroneous belief arose from the fact that the whole intuitionist system corresponds to only a part of the classic results. But Gödel[9] recently found, conversely, that the whole classic calculus of propositions and theory of numbers (including the principle of the excluded middle) correspond to a part of the intuitionist theories, for every classical theorem within this field can be translated into an intuitionist theorem by the use of a simple dictionary. Were there a contradiction between any two classical theorems, there would also be one between the two corresponding intuitionist theorems. One of the rules for translation is that wherever, in the classical theorems, the words "p or q" appear, they must be replaced by the words "it is impossible that p is impossible and q is impossible." Hence, abandoning the principle of the excluded middle, but at the same time admitting (as intuitionists do) the impossibility of universal propositions, does not really limit but only renames the classical theorems in the calculus of propositions and the theory of numbers.

For the special concepts of the theory of sets, I some years ago devised a dictionary[10] by the use of which all the intuitionist concepts within this field may be translated into known special concepts of the classical theory. What Brouwer calls "sets," "individualized sets" and "species" correspond respectively to

[9] See Gödel in "Ergebnisse eines mathematischen Kolloquiums," *4*, 34, Deuticke, Vienna (1933).

[10] Jahresbericht d. deutschen Mathematiker-Vereinigung, *37*, 213 (1928).

"analytic sets," "Borelian sets" and "general subsets of analytic subsets" in the classical theory. Here the difference between the intuitionist and the classical developments lies only in the fact that the intuitionist dogmatically limits himself to the above mentioned special concepts. These he calls meaningful and constructive; more far-reaching ideas he designates as meaningless. But there is no proof of the impossibility of contradictions within the restricted theory. The only certainty is that it excludes large parts of classical mathematics.

How can classical mathematics be defended against these intuitionist attacks? This can be done by a sober analysis of intuitionism. Here two things are to be distinguished: the mental activity of the intuitionist mathematicians, and their verbal and written reports of this activity. The intuitionists, indeed, take the attitude that mathematics consists only of their own constructive mental activity, whereas all external communications are merely more or less imperfect directions for the repetition of this mental activity by other human beings. Naturally, a sober critic can do nothing but stick to their external communications. These are, in part, of a mathematical nature, and contain not only constructions but also proofs which employ certain methods of inference in a perfectly regular fashion; in part they belong rather to a theory of knowledge, and maintain that intuitionist ideas have meaning, whereas more far reaching methods of inference are meaningless.

First, as to assertions like "Certain methods of inference are based on intuition; others are meaningless." Such assertions are, in my opinion, nothing more than descriptions of subjective psychological processes or expressions of subjective tastes. They are therefore of interest only to biographers and historians; they do not belong to logic or mathematics. They are value judgments and thus at root expressions of feeling. According to one's own feelings they may be accepted or opposed by voluntary decisions concerning the methods of inference and construction actually to be used or not to be used. To my mind they recall the dietetic rules enunciated by certain philosophical schools.

Secondly, as to the residue of the intuitionist constructions when they have been purified and freed from their non-mathematical elements. This residue consists of the deduction of propositions by the help of certain methods and modes of inference; it is a system of inferences according to certain rules. But classical mathematics is nothing else. Intuitionists, to be sure, say that the whole of their mental constructions cannot possibly be formulated—cannot possibly be precisely stated in any system of axioms. Yet the inferences which the intuitionists, according to their verbal communications, actually draw, they draw with perfect regularity. These inferences can therefore be collected and traced back to a certain few principles. In fact, Heyting[11] has actually set up a system of axioms for this intuitionist calculus of propositions and of functions. Consequently, the claim of the intuitionists can only mean that they reserve the right to use other inferences besides those contained in any system of axioms. At any given moment, however, the inferences actually drawn by them can be formulated. For example, up to now, in the calculus of propositions and functions, they have drawn no inference which is not permitted by Heyting's system. But should they, because of new discoveries or changes in conviction, use additional methods of inference, then it would merely be necessary to expand the systems of axioms, and again a formal system representing what had actually occurred would be obtained. In intuitionism, what is formulated is, so to speak, limited at the bottom—that is, a certain minimum is assured; it is left open at the top—that is, no maximum is fixed. But similarly, classical mathematics and logic do not by any means insist on the complete immutability of their underlying axioms, and they too reserve the right to introduce new ones. From this point of view, the only difference between the two schools is that intuitionists, although they do not fix a totality of the modes of reasoning, a priori exclude certain modes for all time. In the sense of the simile used above, their dietetic prescription does not

[11] Heyting, Sitzungsberichte d. preussischen Akademie der Wissenschaften, pp. 42, 57, 158, (1930).

enumerate all proper foods. It consists of examples of things to be eaten, together with the prohibition of certain forms of nutriment.

The situation may be elucidated by considering the concepts "definability," "constructability," and "provability" which are discussed in intuitionism. All the applications of these terms and all the rules regarding them have not yet been precisely fixed, not even in Heyting's system. The insistence of the intuitionists on their own constructions and their renunciation of more far-reaching "unconstructive" arguments would seem to me of mathematical importance, particularly in the theory of sets, on one condition—that is, if there were only one way of defining the conditions for constructability, so that the mathematics meeting these conditions would be distinguished from all other possible mathematics. Such however, is certainly not the fact. The postulates for constructability may be varied by degrees. All the imaginable postulates may be combined in such varied ways that it is not even possible to arrange the different combinations in a linear order. There are systems which partly overlap one another. Incidentally the same is true of diets. One code forbids flesh, a more restrictive one interdicts eggs also; a third, overlapping the second, permits eggs but neither flesh nor fish, etc. Observation shows that intuitionists sometimes differ as to what they allow and what they forbid, and even the same intuitionist may change his mind from time to time. Compare the theory of Lusin (based upon the views of Borel, Lebesgue, et al.) with that of Brouwer, or contrast the various stages of Brouwer with one another. There is no question that these various theories have a certain interest, but their interpretation by their respective authors is, I believe, untenable.

What seems to me of importance, what I have for years tried to point out, and what I have attempted to sum up in the foregoing remarks are the following facts:

(1) All that any intuitionist does is to realize one out of a great variety of possible conditions which can partially be arranged in order of stringency, but which partially overlap one another.

(2) Were the intuitionist to omit the statement that his theory is the one founded on intuition whereas all more far-reaching arguments transcend intuition, he would lose no mathematical advantage. Such dicta are only weak points open to attack, for they introduce nothing but psychological elements, personal attitudes and value judgments.

(3) It is certainly possible to construct mathematical systems more restricted than any one so far developed. The negation of universal propositions might be excluded, or, after elementary steps of control had been defined, only conclusions open to control by a limited number (say a million) steps might be admitted. Perhaps some one will some day succeed in stating precisely what is meant by a finitistic mathematics; for this task, in spite of much talk, has not yet, so far as I can see, been accomplished. Another possibility would be the development on the basis of modern logic, of a mathematics suitable for physics, a problem of great importance for this latter science.

The third program followed after the discovery of the antinomies consists in the development of the formalistic or metamathematical method of Hilbert.[12] The thought underlying this method may be condensed as follows: For every mathematical or logical theory, it must first be stated how the undefined fundamental concepts of the theory are symbolized, and how the propositions of the theory are constructed as sets of these fundamental symbols. For instance, there exists an axiomatic representation of Euclidean geometry in which the fundamental concepts are called points, lines and planes. One of the fundamental relations is that of "lying on." One of the rules by which geometric propositions are built up from these fundamental symbols is that, wherever the symbol for a point precedes the

[12] Easily comprehended expositions: Bernays, Blätter f. deutsche Philosophie, *4*, 326, (1936); Herbrand, Revue de metaphysique et de Morale *37*, 243 (1930); Hilbert "Die Grundlagen der Mathematik," Hamburger Mathematische Einzelschriften, vol. 5, Teubner, Leipzig (1928); von Neumann, Erkenntnis, *2*, 116 (1931).

words "lies on," the symbol for a line or a plane follows these words. An example of a logical theory, the axiomatics of the calculus of propositions, has already been given. The fundamental concepts are called "propositions" and denoted by the letters p, q, r, etc. The undefined fundamental relations are "not" and "implies." Secondly, the axioms of the theory must be formulated; that is, the sets of symbols corresponding to certain propositions are set down in advance, as initial formulae. And lastly, there must be given the formative rules whereby new sets of symbols may be deduced from sets which correspond to propositions of the theory. The propositions corresponding to these new sets are then taken into the theory. In the axiomatics of the calculus of propositions there were three such formulae and two such rules (see p. 305).

Thus the theory becomes a calculus, and the theory of this calculus is called the metatheory belonging to the original theory. This metatheory deals with the way in which the propositions of the original theory are connected, and how they may be derived from one another. It considers what propositions can be proved or refuted from the axioms, etc. The metatheory for the axiomatics of the calculus of propositions deals among other things with the question: What propositional formulae can be obtained by applying the two formative rules to the three axioms (initial formulae)? In the axiomatic Euclidean geometry, it is proved, for example, that the sum of the angles in any triangle is equal to 180°. That is, this theorem is deduced from the axioms by the help of the formative rules. On the other hand, metageometry investigates the question which of the Euclidean axioms are necessary for the deduction of this theorem. The proof that the axiom of parallels is independent of the other axioms is metageometrical.

The metatheory belonging to a given theory is above all expected to prove that the theory contains no contradiction—in other words that, in the theory in question, it is never possible to prove an assertion and its negative. By metageometrical considerations it is proved, that, if Euclidean geometry is free from contradiction, then non-Euclidean geometry is also free.

Moreover, it has been possible to show by metamathematical considerations that, if the theory of real numbers is free from contradiction, then Euclidean geometry is likewise. Now Hilbert hoped to prove by metalogic or metamathematics that a logic or mathematics founded on suitable axioms is also free from contradiction. In this way he hoped not only to get logic and mathematics over the crisis caused by the actual discovery of the antinomies, but also to make these subjects secure for all time by the valid proof that within them any sort of contradiction is impossible.

Before going into what has been brought to light by the use of metamathematics,[13] it is well to mention a few general results which are of interest in this connection. In geometry, beside Euclidean geometry, there have been deduced from other axioms other geometries which are quite different from one another and of which each is a system closed within itself. Similarly there have been constructed numerous logics which differ from one another; and each of these is a system closed within itself. Some examples are the so-called polyvalued logics which originated with Lukasiewisz and Post.[14] In the ordinary logic, all propositions are divided into two classes, the class of so-called true and the class of so-called false propositions, so that every proposition belongs to one and only one of the two classes, as expressed by the principle of the excluded middle or third. Likewise, there has now been developed a logic in which propositions are divided into three classes, and a principle of excluded fourth holds. The assumption that through one point there are many parallels to a given line leads to a system which is not only abstract but can even be illustrated by models; similarly, it is possible to get an illustration of the three-valued logic by dividing propositions into surely true, uncertain and surely false ones. In general, there is, for every natural number n, an n-valued logic which

[13] Metamathematics itself has recently been developed as a deductive theory in some important papers by Tarski, Monatashefte f. Mathematik u. Physik, *37*, 361 (1930); Fundamenta Mathematica *25*, 503 (1935) and *26*, 283 (1936).

[14] Post, American Journal of Mathematics *43*, 163 (1921); Lukasiewicz, Compt. Rend. Soc. d. Sciences et d. Lettres Warsaw, *23*, 51 (1930); Lukasiewicz and Tarski, ibid, p. 1. See also the older literature there quoted.

divides propositions into n classes and contains a principle of excluded $n + 1$. The ordinary logic with its dichotomy of propositions comes under this classification as a two-valued logic. To each one of these n-valued logics, there belongs a mathematics; but, of these, only the one associated with the two-valued logic has been closely studied. However, the mere existence of different logics and of different mathematics belonging to them is of interest, for until recently any such state of affairs was regarded as out of the question. It was, in fact, explicitly designated by Poincaré as impossible. Apart from this, these many-valued logics may well be related to the theory of probability.

Another variant from the classical system of logistics is Lewis's[15] theory of what he calls "strict implication." This term is applied to a relation between two propositions which comes nearer to the ordinary use of the word "implication" than the terminology of the calculus of propositions. It thus avoids certain paradoxical consequences of the latter, such as "a false proposition implies every proposition" or "a true proposition is implied by every proposition."

If the problem in hand is to prove by a metatheory that a certain theory contains no contradiction, it must above all be clear what means of proof are permitted for the metatheoretical considerations. If any geometrical or other special theory is to be proved free from contradiction, all of logic can, if necessary, be used as a metatheoretical tool, so that the upshot of the whole argument is a proof by logical inference that, in the system of propositions of the theory in question, there is no contradiction—no assertion appears with its negative. What is meant, however, by a proof that logic itself and the mathematics involved in general logic are free from contradiction? Here, in a proof of self-consistency, not all of logical inference can be used in the metatheory, otherwise the very system whose self-consistency was to be proved would be used as a means of proof. The object is naturally to prove self-consistency for as large a part as possible of logic and mathematics by the use of a minimum of metalogical methods of proof, and eventually in this way to cover all parts of

[15] Cf. Lewis and Langford's book quoted in (1).

the subject. For example, Herbrand succeeded in proving the self-consistency of the calculus of propositions and a part of the theory of natural numbers (containing the principle of the excluded middle) by the use of complete induction, but without the use of the principle of the excluded middle.

The program was to push forward to a proof of self-consistency for all of mathematics. Considering that a portion of mathematics is to be used in this proof, the fundamental problem may be formulated as follows: To prove the self-consistency of all logic and mathematics by the use of a part of logic and mathematics. Such was the state of science until a short while ago, in fact until 1930, when Gödel succeeded in making a completely unexpected and most significant discovery. He solved the fundamental problem, but in a negative sense, for he proved metalogically by the use of only the theory of natural numbers that the self-consistency of mathematics and logic cannot be proved by a part of mathematics and logic.[16] And this conclusion is not the result of a flaw in the system of logical axioms, after the correction of which logic might be proved self-consistent, for Gödel proved the general theorem: Any formal theory which contains the theory of the natural numbers cannot be proved self-consistent by means of principles which can possibly be expressed within the theory in question. No matter how the system of logic is modified, provided it remains inclusive enough to serve as a foundation for the theory of the natural numbers, it still cannot be proved self-consistent by methods which can be expressed within the system. If there are added certain other metalogical methods of proof, which cannot be formulated within the logical-mathematical theory to be proved self-consistent, then the self-consistency of the theory under consideration can be proved; but then the methods of proof are, at least in certain respects, more inclusive than the theory whose self-consistency was to be demonstrated. For example, the theory of the natural numbers can be proved self-consistent, if operations with any class of natural numbers are taken for granted—that is, essentially, if operations with real numbers are admitted. What has turned out to be impossible

[16] Cf. Gödel, Monatshefte f. Mathematik u. Physik, *38*, 173 (1930).

is to prove with a portion of mathematics the self-consistency of a more inclusive portion of mathematics, provided the latter contains the theory of the natural numbers. What can be proved self-consistent by a portion of mathematics, is in general only a narrower or overlapping portion of mathematics. In other words, for the proof of the self-consistency of a portion of mathematics, a more inclusive or an overlapping portion of mathematics is necessary.

This result is so fundamental that I should not be surprised if there were shortly to appear philosophically minded non-mathematicians who will say that they never had expected anything else. For it should be clear to a philosopher that no theory, the structure of which contains no superfluous parts, can be founded on one of its parts, etc. etc. But when such general principles are applied to metalogical problems, they turn out to be not only not self-evident but false. For example (as already stated) it can be proved from the axioms of the theory of real numbers, that, from the self-consistency of this theory, there follows the self-consistency of n-dimensional Euclidean and non-Euclidean geometry, although n-dimensional geometry includes the whole theory of the real numbers as a part, in fact as a one-dimensional special case. On the other hand, Gödel's investigation shows, for example, that it cannot be proved from the axioms of the theory of natural numbers (first systematized by Peano) that the self-consistency of the theory of real numbers follows from the self-consistency of the theory of natural numbers. Similarly, it cannot be proved by certain parts of the theory of natural numbers that the self-consistency of the whole theory follows from the self-consistency of these parts. This contrast shows clearly that Gödel's discovery is not a self-evident remark based on general principles, but a fundamental mathematical theorem which needs proof and can be proved.

Quite recently another important result concerning the self-consistency of mathematics has been obtained by Gentzen.[17] He uses in his metamathematics a part of the theory of natural

[17] Gentzen, Mathematische Annalen, *112*, 493 (1936). Transfinite methods in the theory of proofs are also used by Church.

numbers together with certain transfinite methods. The part of arithmetic used includes complete induction but not what Gentzen calls questionable portions of arithmetic. By these means he proves that the whole of arithmetic including the questionable portions just referred to is free from contradiction. In this case a system of axioms leads to a proof of self-consistency for an overlapping system of axioms.

A second part of Gödel's discoveries is another good illustration of the force of metamathematical methods. In order to explain this it is necessary to digress. It was one of Euler's greatest discoveries that theorems concerning the natural numbers can also be proved by so-called transcendental methods—that is by the help of considerations which go beyond natural numbers and the principle of complete induction, since they make use of the concepts limit and continuity as well as of operations with all real numbers and functions. For example, the theorem (discovered by Fermat and proved by elementary means) that every prime number of the form $4n + 1$ can be expressed, and in one way only, as the sum of the squares of two natural numbers has also been proved by transcendental means. No matter how much Euler's discovery was admired (and a branch of mathematics, the so-called analytical theory of numbers has developed out of it), there still persisted the faith that all theorems about natural numbers could be proved by elementary means. For even when elementary theorems were formulated for which only proofs by transcendental means were found, this fact was ascribed to the circumstance that elementary proofs for these theorems had not yet been discovered. But Gödel has proved metamathematically that there are surely theorems and problems about the natural numbers which cannot be proved or solved by elementary means; for their proof or solution transcendental methods must be employed. Among these unsolvable problems are some concerning certain Diophantine equations,[18] such as: Does the equation $P(x_1, x_2, \ldots x) = 0$ where P is a polynomial with integer coefficients admit of integer solutions? Fermat's hypoth-

[18] Cf. Gödel, Ergebnisse eines mathematischen Kolloquiums, *7*, 23 (1936). Cf. also Church, American Journal of Mathematics, *58*, 345 (1936).

esis (previously mentioned) is nothing but the question whether the Diophantine equation P(x, y, z) = 0 has integer solutions when P is the polynomial $x^n + y^n - z^n$ and n is greater than 2. An example of a proposition which can neither be proved nor refuted by elementary means is the following: The Diophantine equation whose insolubility by elementary means was proved by Gödel admits an integer solution. This proposition is of the same type as Fermat's hypothesis. The question of unsolvable problems is thus today in the same state as the theory of transcendental numbers was in the middle of the last century. At that time Liouville had constructed extensive classes of transcendental numbers, but none of the numbers which had been dealt with in other parts of mathematics were known to be transcendental. Only some decades later was it proved that two of the best known numbers (e and π) are actually transcendental. For logic and the theory of knowledge it is of course unimportant that, so far, no classical problem in mathematics has been turned out to be insoluble. The important fact is that there are certainly problems of the same type as those now under discussion which can be formulated and proved to be without solution.

Again this result does not depend on the insufficiency of the special assumptions about the natural numbers. On the contrary, there is a proven general theorem[19] that in every formal theory which includes the whole theory of the natural numbers there occur problems which cannot be solved within the theory in question. Just as there are propositions about natural numbers which can be proved only by methods taken from the theory of real numbers, so there are propositions about real numbers which can be proved only by methods taken from the theory of sets of real numbers. And there are problems about sets of real numbers which can be solved only by assumptions about sets of higher power. In fact, in every formal theory including the whole theory of the natural numbers, there are statements about natural numbers which can not be proved within the theory in ques-

[19] Cf. Gödels paper quoted in (16). Skolem (Norsk. Matem. Forenings Skrifter, p. 73 (1933)) proved that it is impossible to characterize the sequence of natural numbers by a finite number of axioms if the law of the excluded middle is one of them.

tion. In other words, a universal logic (such as Leibniz dreamed of) which, proceeding from certain principles, makes possible the decision of all conceivable questions, cannot exist.

Such is the state of formalism today. What has it accomplished? Evidently a proof of self-consistency for all mathematics and a demonstration of the solubility of all its problems (the original objectives for the sake of which Hilbert founded and developed metamathematics) have not been and, in a certain sense, cannot be attained. Does this mean that a third crisis in logic, a crisis in metamathematics, has occurred? In my opinion, Gödel's results do not show that the formalistic metamathematical method has failed. Only a collateral philosophical interpretation has proved to be one sided and partly wrong. Hilbert's great and undying merit in creating metamathematics lies in the fact that he introduced a precise method for studying and discussing the mutual relations of different mathematical propositions, different methods of logical inference, etc. Truly, the application of this method, particularly in Gödel's hands, has led partly to the destruction of illusions, but it must not be overlooked that the formalistic method alone has made possible an attack in rigorous fashion on problems like self-consistency and solubility. It has replaced vague philosophic talk on such subjects by precise demonstrations. This vigorous research will certainly continue.

Formalism and metamathematics have made it possible to deal generally, systematically and accurately with the question which in my opinion is of supreme interest for mathematicians and logicians: What can be inferred from given hypotheses (precisely formulated systems of propositions) by given methods of inference (precisely formulated systems of rules for the transformation of propositions)? Under this heading come also the intuitionist systems at any given moment, although some intuitionists are not fond of formulating precisely what hypotheses and methods of inference they do at the moment actually use. It can not be repeated too often that various intuitionist systems have been constructed, and that still other and more restricted ones might be. Almost every one of these theories so far invented has claimed to be the one and only intuitive mathematics and has ruled out more far-reaching systems. But, in my opinion, this attitude is a philosophical interpretation unessential for mathematics, since it plays no rôle at all in the system itself. I should treat this interpretation as a physiologist might treat the different diets preached by various philosophical schools. He would be interested; he would discuss and study the consequences of each one of these diets for bodily and mental well-being; he would compare the consequences of different diets, etc. But he would leave to the individual the choice of which diet to adopt. In mathematics the essential fact is that various systems of hypotheses and various systems of rules of inference (rules for the transformation of hypotheses) may be adopted. Anyone is free to choose as he pleases.[20] Precisely what is interesting is to observe the development (that is the investigation of the consequences) of different systems, and to compare them one with another.

In 1932 I tried to sum up this view in the following words.[21] "What interests the mathematician and all that he does is to deduce propositions by certain methods (which must be formulated) from certain initial propositions (which must also be formulated). There is a variety of ways to choose both the methods and the initial propositions. And to my mind all that mathematics and logic can say about this mathematical activity (which neither needs justification nor can be justified) lies in the statement of these simple facts. Which initial propositions and methods of inference the mathematician and logician choose, and what is the relation of these propositions and methods to so-called reality and to an inner feeling of conviction (Evidenzerlebnis)—such questions belong to other and less exact sciences."

[20] For this reason, the frequent statement that mathematics is a great tautology or system of tautologies does not seem to me an adequate description of the situation. The concept of tautology has so far been defined only within the calculus of propositions. This concept might, it is true, be defined by methods lying outside this calculus, but it might be defined in different ways, whereas the words "Mathematics is a system of tautologies," suggest a reference to an absolute logic.

[21] "Die neue Logik." Krise und Neuaufbau in den exakten Wissenschaften. Deuticke, Vienna, 1933.

What I have said so far must not be misunderstood to indicate that I consider mathematics meaningless or merely a game. I do believe, it is true, that mathematics has important esthetic qualities. It may be left to the psychologist to describe them fully. Partly they consist in an enjoyment of a particular kind of mental activity; partly they resemble the receptive appreciation of music. And no one reproaches music with the fact that its human importance is restricted to its esthetic aspects. Unfortunately the appreciation of mathematics is much rarer than that of music; however this seems to be essentially only an effect of bad teaching.

But no matter how many-sided these entertaining aspects of mathematics may be and indeed are, the importance of mathematics is not restricted to them. This fact is most clearly evidenced by geometry, the axioms of which are related to empirical facts about what is called space. A hundred years ago, what could have seemed to the superficial observer more like a mere game than deductions based on the assumption that in a plane there is more than one parallel to a given straight line through a given point? And yet it is known today that the paths of light rays and of mass points under the influence of gravitation behave like the straight lines in these speculative systems which seem at first to be mere games estranged from all reality. Moreover, these curious deductions have drawn attention to a wealth of further relations between entities which can be experienced and observed, while at the same time they have furnished means for a quantitative check on the assumptions involved. Clearly the views on mathematics here outlined by no means imply that geometry is a mere game or that its propositions are unimportant or meaningless. They only express the belief that the specific function of the geometer is not to deal with these applications, important as they are. For practical and historical reasons, applications are better left to the physicist. The geometer himself has work enough to do in proving new theorems, in studying the mutual relations of the theorems in any single system, in comparing different systems and in creating new ones. Theorems in other parts of mathematics are related to other parts of physics, to the theory of heredity, to the theory of human economic behavior, to ethics, etc. The concept of mathematics here described denies neither the possibility nor the great importance of mathematical applications.

There are also relations between certain hypotheses and modes of inference on the one hand and certain rather vague psychological phenomena like inner conviction or intuition on the other. It is important though usually overlooked that such phenomena change with time, and depend upon history and practice. Or is it to be believed that the methods of inference and construction which are self-evident to an intuitionist in 1930 are identical with the methods of inference and construction which were self-evident to Pythagoras, to Archimedes or to Euler, or which will be self-evident to a mathematician in 2930? Again, the relation between certain parts of mathematics and the feeling of inner conviction is important both from an historical and from a psychological point of view. I only question whether it be practical to deal with such problems within mathematics itself. But if they are so dealt with, I would insist that the mathematician sharply separate them from the mathematical theories in which they play no rôle.

But mathematics (and particularly mathematics as described above) besides its esthetic qualities and the importance of its relations and applications to many sciences has another highly important function. It clarifies discussion on any subject. The theme may be philosophical problems, methodological theories or sociological questions—although this last aspect of mathematics has not yet been extensively developed.[22] Yet this clarifying function may turn out to be of considerable importance for mankind. Mathematics might and perhaps will be the source of a new insight into human intellectual affairs, as well as into practical affairs, so far as they involve intellectual elements.

The importance of mathematics beyond its esthetic qualities is in no wise denied by the assertion that mathematics transforms

[22] An attempt in this direction is my paper, "Einige neuere Fortschritte in der exakten Behandlung sozialwissenschaftlicher Probleme." Neuere Fortschritte in den exakten Wissenschaften, Deuticke, Vienna, 1936.

precisely formulated hypotheses according to precisely formulated methods. Restriction of all discussion on the "foundations" of mathematics by this simple statement serves to emphasize the freedom of choice in respect to both hypotheses and methods. Yet this point of view, although it has often been urged since 1927, has so far had no considerable influence on discussion. Still I hope it will have in the future. Intuitionists in particular have paid no attention to the criticism formulated under three heads on page 322. Only quite recently I have had the pleasure of seeing Carnap state his agreement with this point of view. Freedom of choice in regard to hypotheses and methods of inference he has called the "principle of logical tolerance," and in the introduction of his book "Logische Syntax der Sprache" he describes the effect of this principle (which lies close to the center of his whole theory) in overcoming historical barriers and opening up a world of free possibilities.[23]

What then are the conclusions as to the nature of mathematics? In recent years particularly, the profoundest parts of this science have developed in such a way as to inspire with admiration the very persons who best understand fundamental logical relations. On the other hand it has been impossible to make the subject secure against the occurrence of contradictions. I would say: Mathematicians are like men who build houses. These are not only pleasant to live in; they enable their inhabitants to do many

[23] After introducing the term "principle of logical tolerance," Carnap (logische Syntax der Sprache," Springer, Vienna (1934)) suggests that, in first explicitly formulating this principle, I was expressing the attitude of most mathematicians. I should be glad if Carnap were right, but since the prominent mathematicians (Poincaré, the Paris school at the beginning of the century, Hilbert, Weyl, Brouwer) who have dealt with the foundations of mathematics have explicitly expressed opinions which, divergent as they are from one another, are all diametrically opposed to the above mentioned principle, I am afraid that I must bear the responsibility alone.

I take this opportunity to emphasize that I do not agree with the pronouncements on metaphysics which come from the group of which Carnap is a member. It is true that an expert logician can easily find logical errors, and very elementary errors at that, in many metaphysical theories, just as he can in many economic and sociological ones. Such errors are, however, objections only to special existant systems. If all metaphysics is rejected because its statements cannot be tested, then, I am afraid, very extensive parts of mathematics must likewise be discarded. Thus one arrives at arbitrary statements which resemble some of the critiques of classic mathematics discussed in this paper.

things which a cave-dweller could never accomplish. Mathematicians are like men who build, although they are not sure that an earthquake will not destroy their buildings. If an earthquake should destroy their work, new constructions, if possible such as promise to be more resistant to earthquakes, will be erected. But men will never permanently decide to give up the building of houses with all their conveniences because of possible earthquakes—the more so because absolute security against the effects of earthquakes is not to be obtained even by the inconvenient habit of cave-dwelling. Mathematics seems to me to be in the same state. Not only is it a pleasure in itself, but it serves many other important purposes as well. Its various edifices are not secure against the earthquake of a contradiction. But men will not cease on that account to elaborate its constructions and to erect new ones.

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